परमाणु ऊर्जा शिक्षण संस्था

(परमाणु ऊर्जा विभाग का स्वायत्त निकाय, भारत सरकार)

ATOMIC ENERGY EDUCATION SOCIETY

(An autonomous body under Department of Atomic Energy, Govt. of India)

Chapter 6.
Application Of Derivatives

Module-3

e-content

MAXIMA AND MINIMA

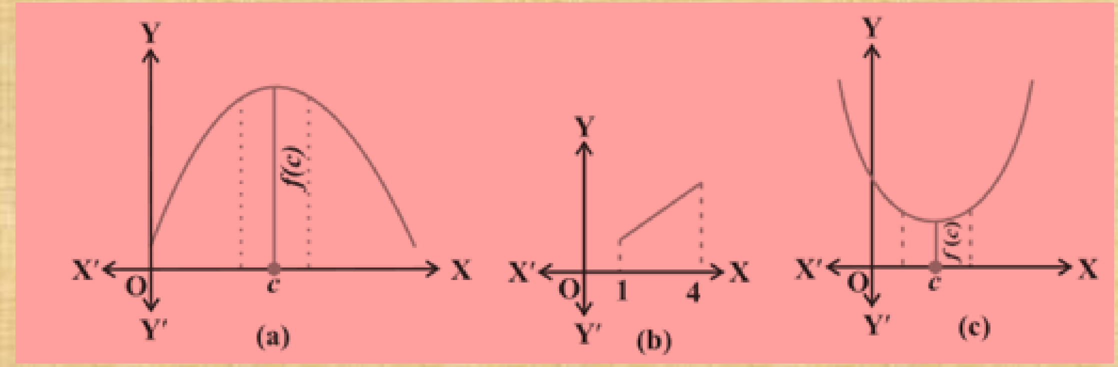
- Let f be a function defined on an interval I. Then
- f is said to have a maximum value in I, if there exists a point c in I such that $f(c) > f(x), for\ all\ x \in I$.
- The number $f(c)$ is called the maximum value in I and the point c is called a point of maximum value of f in I.

- f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x), for\ all\ x \in I$.

- The number $f(c)$ is called the minimum value in I and the point c is called a point of minimum value of f in I.

- f is said to have an extreme value in I, if there exists a point c in I such that
- $f(c)$ is either a maximum value or a minimum value of f in I.

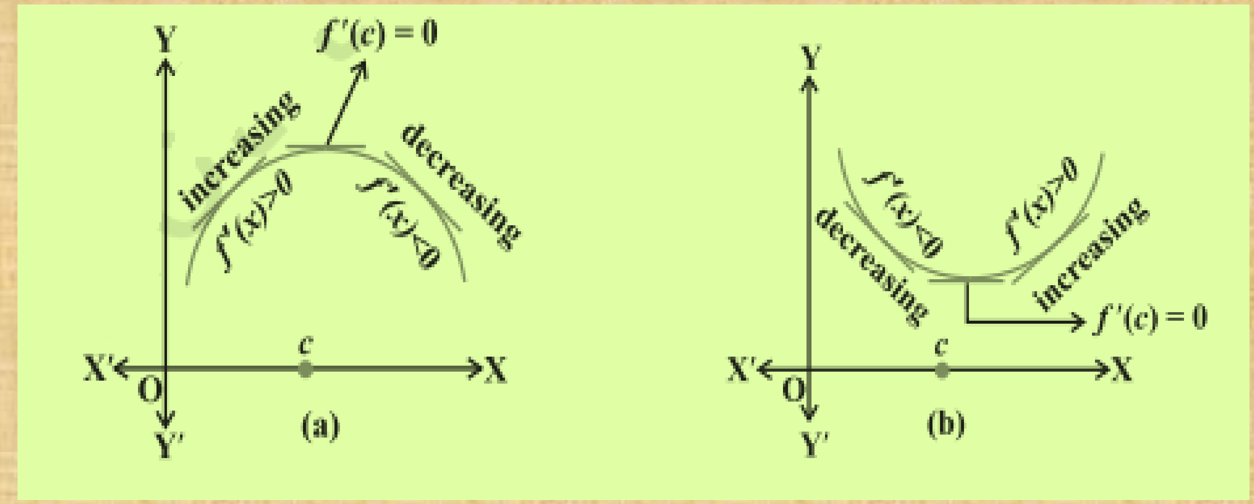
- The number $f(c)$,in this case, is called an extreme value of f in I and the point c is called an extreme point.



LOCAL MAXIMA AND LOCAL MINIMA

- Examine the graph of a function given(next slide). Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called turning points of the given function.
- Further, observe that at turning points, the graph has either a little hill or a little valley. The function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective valleys.
- Similarly, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills.
- For this reason, the points A and C may be regarded as points of local value (or relative minimum value) and points B and D may be regarded as points of local minimum values (or relative maximum value) for the function.
- The local maximum value and local minimum value of the function are referred to as local maxima and local minima, respectively of the function.

- **Definition**:- Let f be a real valued function and let c be an interior point in the domain of f. Then
- c is called a point of local maxima if there is an $h > 0$ such that $f(c) > f(x), for\ all\ x\ in\ (c-h, c+h)$
- The value $f(c)$ is called the local maximum value of f.
- c is called a point of local minima if there is an $h > 0$ such that $f(c) < f(x), for\ all\ x\ in\ (c-h, c+h)$
- The value $f(c)$ is called the local minimum value of f.
-



- Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be shown above in figure (a). Note that the function f is increasing i.e $f'(x) > 0$ in the interval $(c - h, c)$ and decreasing i.e $f'(x) < 0$ in the interval $(c, c + h)$.

- This suggests that $f'(c)$ must be 0.

- Similarly, if c is a point of local minima of f, then the graph of f around c will be shown above in figure (b). Here the function f is decreasing i.e $f'(x) < 0$ in the interval $(c - h, c)$ and increasing i.e $f'(x) > 0$ in the interval $(c, c + h)$.

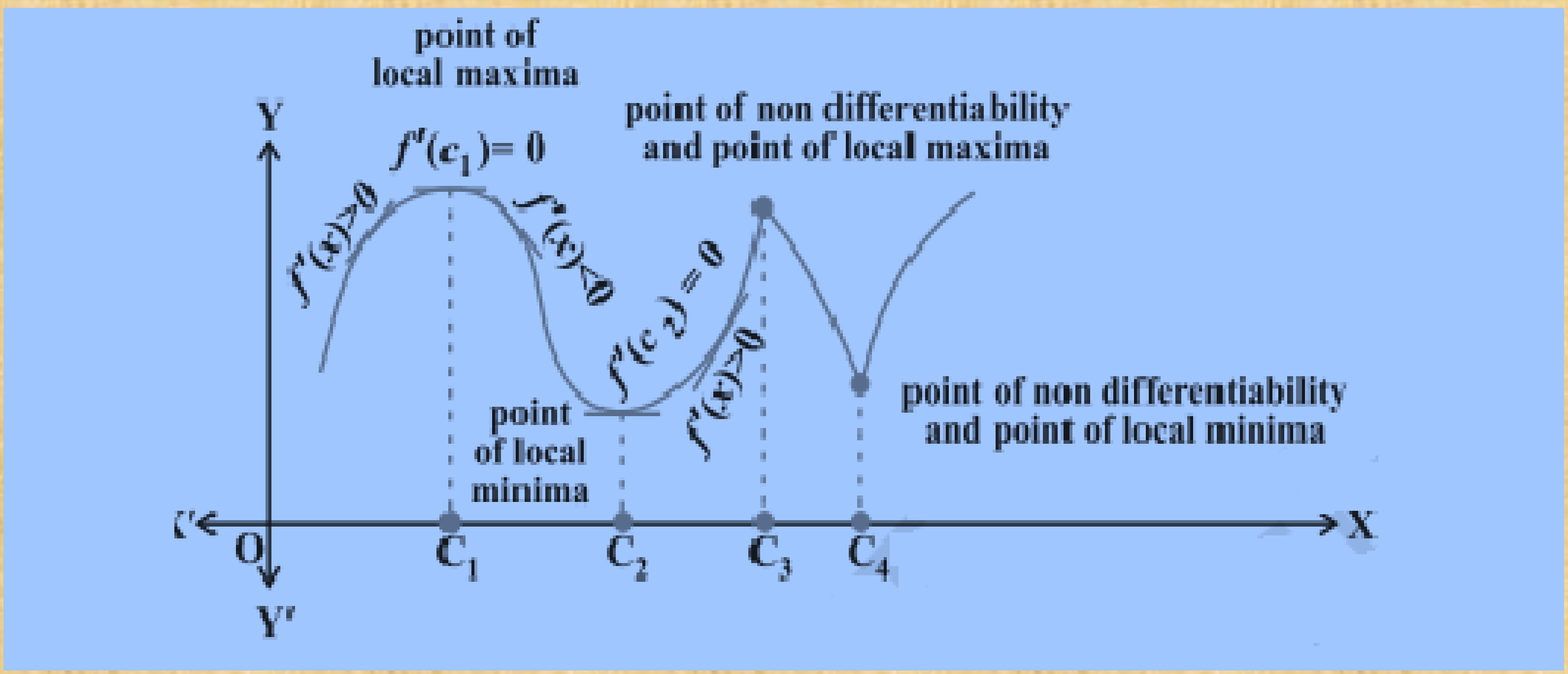
- Again this suggests that $f'(c)$ must be 0.

Theorem (First Derivative Test):- Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

- If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of local maxima.

- If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.

- If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflection.



Theorem (Second Derivative Test):- Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

- $x = c$ is a point of local maxima if $f'(c) = 0\ and f''(c) < 0$
- The value $f(c)$ is local maximum value of f.
- $x = c$ is a point of local minima if $f'(c) = 0\ and f''(c) > 0$
- The value $f(c)$ is local minimum value of f.

- The test fails if $f'(c) = 0\ and\ f''(c) = 0$.

- In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Example 1. Find local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

- **Solution**:- We have, $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.
- $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2)$
  $= 12x(x-1)(x+2)$
- $f'(x) = 0 \quad at\ x = 0, x = 1\ and\ x = -2$
- $f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 1)$
- $f''(0) = -12 < 0, \quad f''(1) = 48 > 0, \quad f''(-2) = 84 > 0$
- Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f $at\ x = 0\ is\ f(0) = 12$.
- While $x = 1\ and\ x = -2$ are the points of local minima and local minimum values of f at $x = -1\ and\ -2\ are\ f(1) = 7\ and\ f(-2) = -20$, respectively.

Maximum and Minimum Values of a Function in a Closed Interval

- Theorem:-Let f be a continuous function on an interval $I = [a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

- Theorem:- Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

- $f'(c) = 0$ if f attains its absolute maximum value at c.

- $f'(c) = 0$ if f attains its absolute minimum value at c.

WORKING RULE:-

- Step 1: Find all critical points of f in the interval, i.e find points x where either $f'(x) = 0$ or f is not differentiable.
- Step 2: Take the end points of the interval.
- Step 3: At all these points , calculate the values of f.
- Step 4: Identify the maximum and minimum values of f out of the values calculated in step 3.
- The maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 2. Find the absolute maximum and minimum values of $f(x) = 2x^3 - 9x^2 + 12x - 5 \ \ in \quad [0,3]$

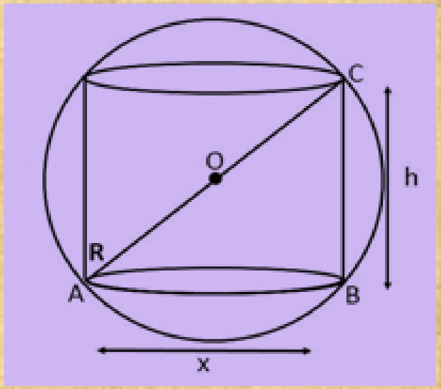
- **Solution**:- Given $f(x) = 2x^3 - 9x^2 + 12x - 5 \ \ in \quad [0,3]$ ....(i)
- It is differentiable for all x in $[0,3]$.
- Differentiating (i) w.r.t.x, we get
- $f'(x) = 2.3x^2 - 9.2x + 12 = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$
-
- Now $f'(x) = 0 \Rightarrow 6(x^2 - 3x + 2) = 0 \Rightarrow x^2 - 3x + 2 = 0.$
- $(x-1)(x-2) = 0 \Rightarrow x = 1,2.$
- Also 1, 2 both are in $[0,3]$, therefore, 1 and 2 both are turning points.
- Further, $f(1) = 2.1^3 - 9.1^2 + 12.1 - 5 = 2 - 9 + 12 - 5 = 0,$
- $f(2) = 2.2^3 - 9.2^2 + 12.2 - 5 = 16 - 36 + 24 - 5 = -1,$
- $f(0) = -5$
- and $f(3) = 2.3^3 - 9.3^2 + 12.3 - 5 = 54 - 81 + 36 - 5 = 4$
- Therefore, the absolute maximum value$= 4$ and the absolute minimum value $= -5$. The point of maxima is 3 and the point of minima is 0.

Example 3. Show that the height of a cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{3}$.
Also find the maximum volume.

- **Solution:-** Let h be the height and x be the diameter of the inscribed cylinder, then

$h^2 + x^2 = (2R)^2 \Rightarrow h^2 + x^2 = 4R^2 \;\; x^2 = 4R^2 - h^2$…………..(i)

Radius of the cylinder $= \frac{x}{2}$



Let V be the Volume of the inscribed cylinder,

Then $V = \pi\left(\frac{x}{2}\right)^2 . h = \frac{\pi}{4}x^2 h$

$V = \frac{\pi}{4}(4R^2 - h^2)h \Rightarrow V = \frac{\pi}{4}(4R^2 h - h^3)$ …..(ii) (using (i))

- Differentiating (ii) w.r.t.h, we get
- $\frac{dv}{dh} = \frac{\pi}{4}(4R^2.1 - 3h^2)$ *and* $\frac{d^2v}{dh^2} = \frac{\pi}{4}(0 - 6h) = -\frac{3}{2}\pi h.$
- Now $\frac{dv}{dh} = 0 \Rightarrow \frac{\pi}{4}(4R^2 - 3h^2) = 0 \Rightarrow 3h^2 = 4R^2$
- $\Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow \qquad h = \frac{2R}{\sqrt{3}} \qquad (h > 0)$
- Also when $h = \frac{2R}{\sqrt{3}}$, $\frac{d^2v}{dh^2} = -\frac{3}{2}\pi.\frac{2R}{\sqrt{3}} = -\frac{3\pi R}{\sqrt{3}} < 0$
- $\Rightarrow V$ *is maximum when* $h = \frac{2R}{\sqrt{3}}.$
- Also when $h = \frac{2R}{\sqrt{3}}$, $V = \frac{\pi}{4}\left(4R^2 - \frac{4R^2}{3}\right).\frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}.$
- Therefore, the volume of the inscribed cylinder is maximum when the height of cylinder $= \frac{2R}{\sqrt{3}}$ and its maximum volume $= \frac{4\pi R^3}{3\sqrt{3}}.$

THANK YOU

Prepared by
Danavath Krishna
PGT,AECS-1, JADUGUDA